3. VIL'KE V.G., On the evolution of the motion of a heavy symmetrical body carrying visco-plastic rods. Vestn. MGU, Matem. Mekhan., Issue 2, 1982.
4. ARKHANGEL'SKII IU.A., Analytical Dynamics of a Solid. Moscow, NAUKA, 1977.
5. VIL'KE V.G., Separation of the motions and the method of averaging in the mechanics of systems with an infinite number of degrees of freedom. Vestn. MGU, Matem. Mekhan. Issue 5, 1983.
6. MOROZOV V.M., RUBANOVSKII V.N., RUMIANTSEV V.V. and SAMSONOV V.A., On the bifurcation and stability of steady motions of complex mechanical systems. PMM, Vol.37, No.3, 1973.

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ON THE CONTROLLED ROTATION OF A SYSTEM OF TWO RIGID BODIES WITH ELASTIC ELEMENTS*

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The problem of controlling the plane rotational motions of two rigid bodies connected by an elastic rod is studied. One end of the rod is attached to the support by a hinge with a spring, the latter modelling the elastic compliance of the fastening, and the other end is rigidly joined to the load. The Hamilton principle is used to obtain the integrodifferential equations and boundary conditions describing the motion of the system support - spring - rod - load. The following problem is posed: it is required to rotate the system by a given angle by means of the controlling force moment, with quenching of the relative oscillations of the load elements which appear as a result of the deformability of the rod and of the elastic torsion of the spring. Similar problem arise in the study of the dynamics and control of the motion of devices used in transporting loads through space (robots, manipulators, load lifting machines, etc.). In computing their control modes a significant part is played not only by the deformability of the elements /1—3/, but also by the elastic compliance of the connecting joints /4/. Asymptotic methods are used to botain a solution of the control problem in question for two limiting cases: 1) the mass of the load carried is much greater than the mass of the rod and support, and 2) the rod has high flexural rigidity. The results obtained represent a development and generalization of the results obtained in /5/. The problems of the dynamics and control of oscillating systems with distributed parameters were investigated using various types of formulation in a number of papers (/5—13/ et. al.).

1. Description of the model and the equations of motion. We consider a mechanical system consisting of two rigid bodies connected by a rod of variable cross-section. The system can execute rotational motions in some plane (Fig.1). One end of the rod is attached to the support G_1 by means of a hinge with a weightless spring, modelling the elastic compliance of the joint. The other end is rigidly fixed to the load G_2, whose linear dimensions are small compared with the length of the rod. The O_1Z_1-axis, perpendicular to the plane of the motion represents the axis of rotation, with respect to which the moment of control forces $\mathrm{M}(t)$ is applied. We introduce the $OXYZ$ coordinate system with origin at the centre of the hinge (point O), rotating in the inertial $O_1X_1Y_1Z_1$ space together with the spring and rod. We direct the OX axis along the tangent to the neutral line of the rod at the point O, and the OZ axis along the O_1Z_1 axis of rotation. We assume that the motion of the model is described in the framework of the linear theory of thin rectilinear, inextensible

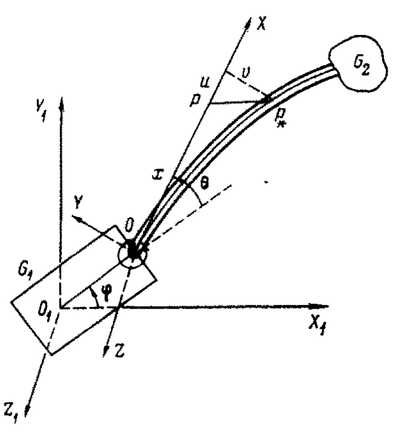

Fig.1

*Prikl.Matem.Mekhan.,48,2,238-246,1984

rods /6, 10/.

We introduce the following notation (Fig.1): x is the abscissa of the point P in the moving OXY $(0 \leqslant x \leqslant l)$ coordinate system where l is the rod length, $\rho(x)$ is the linear density, E is Young's modulus, $I(x)$ is the moment of inertia of the transverse cross-section relative to the axis perpendicular to the plane of flexure, J_1 is the moment of inertia of the body G_1 relative to the axis of rotation, a is the distance between the axis of rotation and the point O, m is the mass of the load G_2, c is the reduced angular coefficient of rigidity of the spring, φ is the angle of rotation of the support body, θ is the additional angular displacement of the rod due to the compliance of the hinge measured from the line O_1O to the OX axis (the angle of twist of the spring $\theta = 0$ corresponds to its stress-free state), $\mathbf{u}(x, t)$ is the displacement vector of the point P of the elastic rod with coordinate x at the instant of time t, $u(x, t)$, $-v(x, t)$ is its projection on the axes of the OXY coordinate system.

The inextensibility of the rod yields the following relation:

$$u_x(x, t) = -{}^1/_2 v_x^2(x, t) \tag{1.1}$$

Here and henceforth the index x denotes the corresponding partial derivative.

Let us derive the equations of motion of the mechanical system in question. Let P denote any point of the rod. The absolute velocity of the point P is

$$\mathbf{V}(x, t) = \mathbf{V}_r + \boldsymbol{\omega}_1 \times \mathbf{r} \tag{1.2}$$

where $\mathbf{V}_r$ is its relative velocity, $\boldsymbol{\omega}_1$ is the angular velocity of the moving coordinate system relative to the inertial frame of reference, and $\mathbf{r}$ is the radius vector of P at the instant t. We have the following coordinate representation (in the linear approximation in θ) for the vectors $\mathbf{r}, \boldsymbol{\omega}_1, \mathbf{V}_r$ in the $OXYZ$ system:

$$\mathbf{r} = O_1\mathbf{P}_* = \begin{Vmatrix} a + x + u \\ -v - a\theta \\ 0 \end{Vmatrix}, \quad \boldsymbol{\omega}_1 = \begin{Vmatrix} 0 \\ 0 \\ \varphi^{\cdot} + \theta^{\cdot} \end{Vmatrix}, \quad V_r = \begin{Vmatrix} u_t \\ -v_t - a\theta^{\cdot} \\ 0 \end{Vmatrix} \tag{1.3}$$

A dot and the subscript t denote, respectively, the total derivative and the partial time derivative.

Using (1.2), (1.3) and eleiminating the longitudinal vibrations $u(x, t)$ with help of relation (1.1) we arrive, after some reduction, at the following expression for the kinetic energy with an accuracy up to and including terms of the second order of smallness:

$$T = \frac{1}{2} J_1 \varphi^{\cdot 2} + \frac{1}{2} \int_0^l \langle \rho(x) \{\varphi^{\cdot 2}(a\theta + v)^2 + [v_t - (x+a)\varphi^{\cdot} - x\theta^{\cdot}]^2\} - \tag{1.4}$$

$$\varphi^{\cdot 2} b(x) v_x^2 \rangle dx + \frac{m}{2} \{\varphi^{\cdot 2} [a\theta + v(l, t)]^2 + [v_t(l, t) - h\varphi^{\cdot} - l\theta^{\cdot}]^2\}$$

Using the same assumptions, we obtain the potential energy of the system

$$\Pi = \frac{1}{2} c\theta^2 + \frac{1}{2} \int_0^l EIv_{xx}^2(x, t)\, dx \tag{1.5}$$

$$b(x) = mh + \int_x^l (a + s)\rho(s)\, ds, \quad h = a + l$$

We derive the equations of motion using the Hamilton principle in the form

$$\int_0^\tau (\delta T - \delta\Pi + \delta^* A)\, dt = 0 \tag{1.6}$$

Here δT, $\delta\Pi$ are the variations in the kinetic and potential energy, $\delta^* A = \mathbf{M}\delta\varphi$ is the elementary work done by the given non-potential forces and $t = 0, t = \tau$ denotes the beginning and end of the control process.

We note that the properties of the $OXYZ$ coordinate system imply the relations

$$v(0, t) = 0, \quad v_x(0, t) = 0 \tag{1.7}$$

Using (1.4), (1.5), (1.7) and integrating by parts (1.6), we obtain a stationary expression, the latter property yielding the following equations and deficient boundary conditions:

$$J\varphi^{\cdot\cdot} + J_2\theta^{\cdot\cdot} - mhv_{tt}(l, t) - \int_0^l \rho(x)(x + a)v_{tt}(x, t)\, dx = M(t) \tag{1.8}$$

$$J_3\theta^{\cdot\cdot} + (c - m_3 a^2\varphi^{\cdot 2})\theta = \int_0^l \rho(x)(xv_{tt} + a\varphi^{\cdot 2}v)\,dx -$$
$$J_2\varphi^{\cdot\cdot} + m[lv_{tt}(l,t) + a\varphi^{\cdot 2}v(l,t)]$$
$$E[I(x)v_{xx}]_{xx} + \rho(x)v_{tt} = \rho(x)[x\theta^{\cdot\cdot} + (x+a)\varphi^{\cdot\cdot}] +$$
$$\varphi^{\cdot 2}\{\rho(x)[v + a\theta - (a+x)v_x] + b(x)v_{xx}\}$$

$$E[I(x)v_{xx}]_x|_{x=l} = m[v_{tt} - h\varphi^{\cdot\cdot} - l\theta^{\cdot\cdot} + \varphi^{\cdot 2}(hv_x - a\theta - v)]|_{x=l} \tag{1.9}$$
$$v_{xx}(l,t) = 0$$

where

$$J = J_1 + mh^2 + \int_0^l \rho(x)(x+a)^2\,dx, \quad m_3 = m + \int_0^l \rho(x)\,dx$$
$$J_2 = mlh + \int_0^l \rho(x)\,x(x+a)\,dx, \quad J_3 = ml^2 + \int_0^l \rho(x)x^2\,dx$$

The first equation of (1.8) expresses the theorem on the change in angular momentum of the whole system relative to the O_1Z_1 axis, the second equation describes the torsion in the hinge spring acted upon by the principal moment of the D'Alembert forces of inertia of the load elements of the system, about the point O (in the linear approximation in θ, $\theta^{\cdot}$), and the third equation describes small elastic displacements of the points of the rod from their equilibrium position. The boundary conditions (1.9) have a dynamic character and express the equilibrium of the transverse force and the fact that there is no bending moment at the end of the rod at $x = l$.

To determine the motion of the system uniquely, we must specify the initial configuration and velocity of the points of the neutral line of the rod for $0 \leqslant x \leqslant l$, and the initial values of φ, $\varphi^{\cdot}$, θ, $\theta^{\cdot}$:

$$v(x,0) = f(x), \quad v_t(x,0) = g(x) \quad (f(0) = f_x(0) = 0) \tag{1.10}$$
$$\varphi(0) = \varphi^\circ, \quad \varphi^{\cdot}(0) = \varphi^{\circ\cdot}; \quad \theta(0) = \theta^\circ, \quad \theta^{\cdot}(0) = \theta^{\circ\cdot} \tag{1.11}$$

Thus the integrodifferential equations in partial derivatives (1.8) with the boundary condition (1.7), (1.9) and initial condition (1.10), (1.11) define uniquely, for the given control $\mathbf{M}(t)$, the motion of the mechanical system in question.

When the hinge O has neither the support G_1, nor a spring, i.e. when $J_1 = a = 0$, $\theta(t) \equiv 0$, (1.7)$\to$(1.9) yield equations and boundary conditions for the controlled rotation of a rod /5/.

We can now formulate the following problem. We require to determine the control $\mathbf{M}(t) \in \mathbf{M}$, which transfers the system, by virtue of (1.8) and boundary conditions (1.7), (1.9), from its initial state (1.10), (1.11) to the final state, with the relative displacements quenched

$$\varphi(\tau) = \varphi_*, \quad \varphi^{\cdot}(\tau) = \theta(\tau) = \theta^{\cdot}(\tau) = 0 \tag{1.12}$$
$$v(x,\tau) = v_t(x,\tau) \equiv 0, \quad x \in [0, l] \tag{1.13}$$

Here $\mathbf{M}$ is a given fixed set of admissible values of the control.

The relations (1.7)–(1.9) imply that the system will remain at rest (1.12), (1.13) for $t > \tau$, provided that we then put $\mathbf{M} \equiv 0$.

2. The problem of control in the quasistatic approximation. To study the problem in question, it is convenient to change to dimensionless variables and parameters. We introduce them in the following manner:

$$t' = \nu t, \quad x' = x/l, \quad v' = v/l, \quad I' = I/I_0 \tag{2.1}$$
$$\rho' = \rho/\rho_0, \quad c' = c/(ml^2\nu^2), \quad a' = a/l.$$

Here ν is the characteristic constant with dimensions of frequency, whose choice is governed by the specific features of the problem, and I_0, ρ_0 are the characteristic parameters of the problem, with the corresponding dimensions of inertia and linear density respectively.

We shall assume that the mass of the body G_1 and the rod is vanishingly small compared with the mass of the load G_2. Then, putting $\mathbf{M}' = \mathbf{M}/(mlh\nu^2)$ and choosing as ν the quantity $\nu = (EI_0/(ml^3))^{1/2}$ characterising the frequency of the quasistatic oscillations of the system, we obtain from (1.7)–(1.9) the following relations (the primes are omitted from the new variables):

$$(1+a)\varphi^{\cdot\cdot} + \theta^{\cdot\cdot} - v_{tt}(1,t) = \mathrm{M}(t) \tag{2.2}$$
$$\theta^{\cdot\cdot} + (c - a^2\varphi^{\cdot 2})\theta = v_{tt}(1,t) - (1+a)\varphi^{\cdot\cdot} + a\varphi^{\cdot 2}v(1,t) \tag{2.3}$$
$$[I(x)v_{xx}]_{xx} = (1+a)\varphi^{\cdot 2}v_{xx} \tag{2.4}$$

$$v(0,t) = v_x(0,t) = v_{xx}(1,t) \equiv 0 \tag{2.5}$$
$$[I(x)\,v_{xx}]_x\,|_{x=1} = \{v_{tt} - (1+a)\,\varphi^{\cdot\cdot} - \theta^{\cdot\cdot} + \varphi^{\cdot 2}\,[(1+a)\,v_x - v - a\theta]\}\,|_{x=1}$$

In the present formulation the initial distribution of the points of the weightless rod $(0 < x < 1)$ and their velocities are not essential for the further motion of the load G_2, and we therefore have

$$v(1,0) = f(1),\quad v_t(1,0) = g(1) \tag{2.6}$$
$$v(1,\tau) = v_t(1,\tau) = 0 \tag{2.7}$$

The formulation of the initial and final conditions for $\varphi, \varphi^{\cdot}, \theta, \theta^{\cdot}$ remains unchanged and is given by relations (1.11), (1.12).

Let us restrict ourselves to the case of $I(x) = I_0 = \text{const}$, which is important in practice. Then taking (2.2) into account, we can write the solution for the boundary value problem (2.4), (2.5) in the form

$$v(x,t) = [\mathrm{M}(t) + a\theta(t)\varphi^{\cdot 2}(t)]\,\Phi(x,t) \tag{2.8}$$
$$\Phi(x,t) = \frac{\operatorname{th}\omega\,(\operatorname{ch}\omega x - 1) - \operatorname{sh}\omega x + \omega x}{\varphi^{\cdot 2}(t)\,(a\omega + \operatorname{th}\omega)},\quad \omega = \varphi^{\cdot}(t)\sqrt{\frac{1+a}{I_0}}$$

Let us integrate (2.2) term by term, taking into account the initial conditions (1.11), (2.6)

$$(1+a)(\varphi^{\cdot} - \varphi^{\circ\cdot}) + \theta^{\cdot} - \theta^{\circ\cdot} - v_t(1,t) + g(1) = \int_0^t \mathrm{M}(s)\,ds \tag{2.9}$$
$$(1+a)(\varphi - \varphi^{\circ}) + \theta - \theta^{\circ} - [\varphi^{\circ\cdot}(1+a) + \theta^{\circ\cdot} - g(1)]\,t - v(1,t) + f(1) = \int_0^t (t-s)\,\mathrm{M}(s)\,ds$$

Equation (2.3) taking (2.2) into account, can be written in the form

$$(c - a^2\varphi^{\cdot 2})\,\theta = a\varphi^{\cdot 2}v\,(1,t) - \mathrm{M}(t) \tag{2.10}$$

Relations (2.8)–(2.10) yield the solution of the problem of control in the quasistatic approximation, namely, the sufficient and necessary condition for the motion of the system determined by the equations (2.2)–(2.4), and the boundary conditions (2.5), and the initial conditions (1.11), (2.6) which satisfy the final conditions (1.12), (2.7), is, that the twice differentiable function $\mathrm{M}(t)$ be chosen from the relations

$$\int_0^\tau \mathrm{M}(t)\,dt = g(1) - \theta^{\circ\cdot} - (1+a)\,\varphi^{\circ\cdot},\quad \mathrm{M}(\tau) = 0 \tag{2.11}$$
$$\int_0^\tau (\tau - t)\,\mathrm{M}(t)\,dt = f(1) + (1+a)(\varphi_* - \varphi^{\circ}) - \theta^{\circ} - [\varphi^{\circ\cdot}(1+a) + \theta^{\circ\cdot} - g(1)]\,\tau$$
$$\mathrm{M}(0) = f(1)\,\Phi^{-1}(1,0) - a\theta^{\circ}\varphi^{\cdot\circ 2},\quad \mathrm{M}^{\cdot}(\tau) = 0$$
$$\mathrm{M}^{\cdot}(0) = [g(1) - f(1)\,\Phi^{-1}(1,0)\,\Phi^{\cdot}(1,0)]\,\Phi^{-1}(1,0) - a\varphi^{\circ\cdot}(\theta^{\circ\cdot}\varphi^{\circ\cdot} + 2\theta^{\circ}\varphi^{\circ\cdot\cdot})$$

Note that the initial conditions for the twist of the spring and for the state of the points of elastic rod are chosen, taking the specific constraints into account. For example, when $\varphi^{\circ\cdot} = \theta^{\circ\cdot} = 0$ the condition of quasistatic equilibrium of the spring and rod with the load G_2, of the form $3I_0 f(1) + c\theta^{\circ} = 0$, must hold. When $a = \theta^{\circ} = \theta^{\circ\cdot} = 0$, the solution of the problem of controlling the rotation of the elastic rod with a laod follows, in the quasistatic approximation /5/, from (2.11).

Let us consider the important practical special case of zero initial conditions $\theta^{\circ} = \theta^{\circ\cdot} = f(1) = g(1) = \varphi^{\circ\cdot} = 0$. Here, as (2.11) implies, the following relations must hold:

$$\int_0^\tau \mathrm{M}(t)\,dt = 0,\ \mathrm{M}(0) = \mathrm{M}^{\cdot}(0) = \mathrm{M}(\tau) = \mathrm{M}^{\cdot}(\tau) = 0 \tag{2.12}$$
$$\int_0^\tau (\tau - t)\,\mathrm{M}(t)\,dt = (a+1)(\varphi_* - \varphi^{\circ}) \tag{2.13}$$

Let the hinge O contain neither the support G_1, nor the spring, and let the rod be rigidly fixed at the point O_1. Then, to rotate the loaded rod by an angle φ_*, with quenching of the relative deflections of the load, the controlling moment $M_1(t)$ must be chosen from conditions analogous to (2.12) and the relation /5/

$$\int_0^\tau (\tau - t)\, \mathrm{M}_1(t)\, dt = \varphi_* - \varphi^\circ \tag{2.14}$$

Taking (2.14) into account we obtain, from (2.13),

$$\int_0^\tau (\tau - t)\, [\mathrm{M}(t) - \mathrm{M}_1(t)]\, dt = a(\varphi_* - \varphi^\circ) \tag{2.15}$$

Since $a(\varphi_* - \varphi^\circ) > 0$, from (2.15) it follows that a time interval $[t_1, t_2]$, $0 < t_1 < t_2 < \tau$ exists in which $M(t) > M_1(t)$.

Thus to bring the mechanical system in question into the prescribed angular position, taking the elastic compliance of the connecting hinge into account, requires an increase in the controlling moment only in the case, when the hinge lies outside the axis of rotation ($a > 0$). If the hinge lies on the axis of rotation ($a = 0$), then the same control force moment chosen from the relations (2.12), (2.14) will bring the loaded rod into the prescribed angular position in the case of a rigid hinge /5/, as well as in the case when the hinge exhibits a concentrated elastic compliance.

3. Investigation of the problem of control in the case of large flexural rigidity of the rod. We choose here $v = (\mathrm{M}_0/J)^{1/2}$ as v, and introduce a new control $\mathrm{M}' = \mathrm{M}/\mathrm{M}_0$, where $\mathrm{M}_0 = \sup_t |\mathrm{M}(t)|$. Then equations (1.8) in the new variables (2.1) will take the form

$$\varphi^{\cdot\cdot} + \frac{J_2}{J}\theta^{\cdot\cdot} - \chi v_{tt}(1, t) - \varkappa \int_0^1 \rho(x)(x + a)\, v_{tt}(x, t)\, dx = \mathrm{M}(t) \tag{3.1}$$

$$\frac{J_2}{J}\theta^{\cdot\cdot} + (c\chi_1 - a^2\chi_2\varphi^{\cdot 2})\,\theta = \varkappa \int_0^1 \rho(x)(xv_{tt} + a\varphi^{\cdot 2}v)\, dx -$$

$$\frac{J_2}{J}\varphi^{\cdot\cdot} + \chi_1[v_{tt}(1, t) + a\varphi^{\cdot 2}v(1, t)]$$

$$[I(x)\, v_{xx}]_{xx} + \mu\varkappa\rho v_{tt} = \mu\varkappa\rho\,\{x\theta^{\cdot\cdot} + (x + a)\,\varphi^{\cdot\cdot} +$$

$$\varphi^{\cdot 2}[v + a\theta - (a + x)\, v_x]\} + \mu\varphi^{\cdot 2}v_{xx}\left[\chi + \varkappa\int_x^1 \rho(s)(a + s)\, ds\right]$$

$$\varkappa = \frac{\rho_0 l^3}{J}, \quad \chi = \frac{mhl}{J}, \quad \chi_1 = \frac{ml^2}{J}, \quad \chi_2 = \frac{m_2 l^2}{J}, \quad \mu = \frac{M_0 l}{EI_0} \ll 1$$

The boundary and initial conditions are

$$v(0, t) = v_x(0, t) = v_{xx}(1, t) \equiv 0 \tag{3.2}$$

$$[I(x)v_{xx}]_x|_{x=1} = \mu\chi_1\{v_{tt} - (1 + a)\varphi^{\cdot\cdot} - \theta^{\cdot\cdot} + \varphi^{\cdot 2}[(1 + a)v_x - a\theta - v]\}|_{x=1}$$

$$\varphi(0) = \varphi^\circ,\ \varphi^{\cdot}(0) = \varphi^{\circ\cdot},\ \theta(0) = \mu\theta^\circ,\ \theta^{\cdot}(0) = \mu\theta^{\circ\cdot} \tag{3.3}$$

$$v(x, 0) = \mu f(x), \quad v_t(x, 0) = \mu g(x)$$

The conditions at the completion of the control process are given by (1.12), (1.13) with $l = 1$.

We will construct the solution of the control problem, using the methods of perturbation theory, in powers of the small parameter μ. We assume that

$$\mathbf{q} = \mathbf{q}_0 + \mu\mathbf{q}_1 + \mu^2 \ldots\ (\mathbf{q} = (\varphi(t), \theta(t), v(x, t), \mathrm{M}(t))) \tag{3.4}$$

When the flexural rigidity of the rod is infinitely large ($\mu = 0$), the points of the rod undergo no relative displacements, i.e. $v_0(x, t) \equiv 0$. The motion of the system as a whole, taking into account the only possible twist of the spring, is described by virtue of (3.1)–(3.3) by the following equations, and initial and final conditions (to terms of the order of $\varphi^{\cdot 2}\theta$):

$$\varphi_0^{\cdot\cdot} + \frac{J_2}{J}\theta_0^{\cdot\cdot} = \mathrm{M}_0(t), \quad \frac{J_2}{J}\theta_0^{\cdot\cdot} + c\chi_1\theta_0 = -\frac{J_2}{J}\varphi_0^{\cdot\cdot} \tag{3.5}$$

$$\varphi_0(0) = \varphi^\circ, \quad \varphi_0^{\cdot}(0) = \varphi^{\circ\cdot}, \quad \theta_0(0) = \theta_0^{\cdot}(0) = 0 \tag{3.6}$$

$$\varphi_0(\tau) = \varphi_*, \quad \varphi_0^{\cdot}(\tau) = \theta_0(\tau) = \theta_0^{\cdot}(\tau) = 0 \tag{3.7}$$

It can be shown that if the motion of the system is determined by (3.5) and initial conditions (3.6), then the purpose of the motion will be achieved if and only if the control $\mathrm{M}_0(t)$ is chosen from the relation

$$\int_0^\tau \mathrm{M}_0(t) \sin k(\tau - t)\, dt = 0, \quad \int_0^\tau \mathrm{M}_0(t) \cos k(\tau - t)\, dt = 0 \tag{3.8}$$

$$\int_0^\tau M_0(t)\,dt = -\varphi^{\circ\cdot}, \quad \int_0^\tau (\tau - t) M_0(t)\,dt = \varphi_* - \varphi^\circ - \varphi^{\circ\cdot}\tau$$

$$k^2 = c\chi_1 J_2/(JJ_3 - J_2^2)$$

From (3.1)–(3.4) we obtain the following relations (to terms $\varphi_1^{\cdot 2}\theta_1, \varphi_1^{\cdot 2}v_1$) for determining the functions $\varphi_1, \theta_1, v_1, M_1$:

$$\varphi_1^{\cdot\cdot} + \frac{J_2}{J}\theta_1^{\cdot\cdot} - \chi v_{1tt}(1,t) - \varkappa \int_0^1 \rho(x)(x+a)v_{1tt}(x,t)\,dx = M_1(t) \tag{3.9}$$

$$\frac{J_3}{J}\theta_1^{\cdot\cdot} + c\chi_1\theta_1 = \varkappa \int_0^1 \rho(x) x v_{1tt}\,dx - \frac{J_2}{J}\varphi_1^{\cdot\cdot} + \chi_1 v_{1tt}(1,t) \tag{3.10}$$

$$[I(x)v_{1xx}]_{xx} = \varkappa\rho[x\theta_0^{\cdot\cdot} + (x+a)\varphi_0^{\cdot\cdot}] \tag{3.11}$$

$$v_1(0,t) = v_{1x}(0,t) = v_{1xx}(1,t) \equiv 0 \tag{3.12}$$

$$[I(x)v_{1xx}]_x|_{x=1} = -\chi_1[\theta_0^{\cdot\cdot} + (1+a)\varphi_0^{\cdot\cdot}] \tag{3.13}$$

$$\varphi_1(0) = \varphi_1^{\cdot}(0) = 0, \quad \theta_1(0) = \theta^\circ, \quad \theta_1^{\cdot}(0) = \theta^{\circ\cdot} \tag{3.14}$$

$$v_1(x,0) = f(x), \quad v_{1t}(x,0) = g(x) \tag{3.15}$$

$$\varphi_1(\tau) = \varphi_1^{\cdot}(\tau) = \theta_1(\tau) = \theta_1^{\cdot}(\tau) = 0 \tag{3.16}$$

$$v_1(x,\tau) = v_{1t}(x,\tau) \equiv 0, \quad x \in [0,1] \tag{3.17}$$

The solution of the boundary value problem (3.11)–(3.13) has the form

$$v_1(x,t) = A(x)\theta_0^{\cdot\cdot}(t) + B(x)\varphi_0^{\cdot\cdot}(t) \tag{3.18}$$

$$A(x) = \int_0^x \frac{x-s}{I(s)} n_1(s)\,ds, \quad B(x) = A(x) + \int_0^x \frac{x-s}{I(s)} n_2(s)\,ds$$

$$n_1(x) = \varkappa \int_0^x (x-s)s\rho(s)\,ds + \chi_1(1-x) - x\varkappa_1 + \varkappa_2$$

$$n_2(x) = \varkappa a \int_0^x (x-s)\rho(s)\,ds + a\chi_1(1-x) - x\varkappa_3 + \varkappa_4$$

$$\varkappa_1 = \varkappa \int_0^1 x\rho(x)\,dx, \quad \varkappa_2 = \varkappa \int_0^1 x^2\rho(x)\,dx$$

$$\varkappa_3 = a\varkappa \int_0^1 \rho(x)\,dx, \quad \varkappa_4 = a\varkappa_1$$

Taking (3.5), into account we can write the solution (3.18) in the form

$$v_1(x,t) = M_0(t)A_1(x) + k^2\left[\frac{J_2}{J}B(x) - A(x)\right]\theta_0(t) \tag{3.19}$$

$$A_1(x) = c_1 A(x) + \left(1 - \frac{J_2 c_1}{J}\right)B(x), \quad c_1 = J_2 J/(J_2^2 - JJ_3)$$

The control $M_0(t)$ is such, that $\theta_0(t)$ satisfies the relation (3.7). Therefore, if we demand, in addition, that conditions

$$M_0(\tau) = M_0^{\cdot}(\tau) = 0 \tag{3.20}$$

hold, then from (3.19) it follows that the relative displacements of the points of the rod will be quenched, i.e. conditions (3.17) will be satisfied.

Analysing the solution (3.19), taking (3.6) into account, we find that the initial conditions (3.15) can be satisfied only when the following relations hold:

$$M_0(0) = f(x)A_1^{-1}(x), \quad M_0^{\cdot}(0) = g(x)A_1^{-1}(x) \tag{3.21}$$

Substituting $v_1(x,t)$ obtained into the equations (3.9), (3.10), we obtain the following relations for determining the remaining unknowns φ_1, θ_1, M_1:

$$\varphi_1^{\cdot\cdot} + \frac{J_2}{J}\theta_1^{\cdot\cdot} = M_1(t) + \Phi_1[M_0^{\cdot\cdot}(t), \theta_0^{\cdot\cdot}(t)] \tag{3.22}$$

$$\frac{J_3}{J}\theta_1^{\cdot\cdot} + c\chi_1\theta_1 = \Phi_2[M_0^{\cdot\cdot}(t), \theta_0^{\cdot\cdot}(t)] - \frac{J_2}{J}\varphi_1^{\cdot\cdot}$$

Here Φ_1, Φ_2 are known functions linear in $M_0^{\cdot\cdot}, \theta_0^{\cdot\cdot}$ determinable by virtue of (3.9), (3.10), (3.19). Taking the initial conditions (3.14) into account we obtain, from (3.22),

$$\theta_1(t)=\theta^\circ\cos kt+\frac{\theta^{\circ\cdot}}{k}\sin kt+\frac{1}{k}\int_0^t[c_1M_1(s)+\Phi_3(s)]\sin k(t-s)\,ds \tag{3.23}$$

$$\Phi_3=c_1(J_2\Phi_1-J\Phi_2)/J_2$$

If we now require that conditions (3.16) hold, we obtain, from (3.23), after some reduction, the following expressions for the unknown control $M_1(t)$:

$$\int_0^\tau[c_1M_1(t)+\Phi_3(t)]\sin kt\,dt=k\theta^\circ \tag{3.24}$$

$$\int_0^\tau[c_1M_1(t)+\Phi_3(t)]\cos kt\,dt=-\theta^{\circ\cdot}$$

Integrating the first equation of (3.22) with respect to time and requiring that conditions (3.14), (3.16) hold, we arrive at the following additional relations:

$$\int_0^\tau[M_1(t)+\Phi_1(t)]\,dt=-\frac{J_2}{J}\theta^{\circ\cdot} \tag{3.25}$$

$$\int_0^\tau[M_1(t)+\Phi_1(t)](\tau-t)\,dt=-\frac{J_2}{J}(\theta^\circ+\tau\theta^{\circ\cdot})$$

Thus we have proved the following assertion. Let the motion of a mechanical system be described by (3.1) and the boundary (3.2) and initial (3.3) conditions. Then the necessary and sufficient condition for the motion to satisfy the conditions (1.12), (1.13) (for $l=1$) up to and including terms of the order of μ, at the instant when the control process is completed is, that the control $M(t)=M_0(t)+\mu M_1(t)$ where $M_0(t), M_1(t)$ be chosen from the relations (3.8), (3.20), (3.21), (3.24), (3.25).

If the hinge has neither a support nor a spring, i.e. $J_1=a=\theta(t)\equiv 0$, then the solution obtained yields a solution for the problem of the control of a loaded elastic rod in the case of large flexural rigidity /5/.

REFERENCES

1. CHERNOUS'KO F.L., Dynamics of controlled motions of an elastic manipulator. Izv. Akad. Nauk SSSR, Tekhn. kibernetika, No.5, 1981.
2. AKULENKO L.D., MIKHAILOV S.A. and CHERNOUS'KO F.L., Modelling the dynamics of a manipulator with elastic elements. Izv. Akad. Nauk SSSR, MTT, No.3. 1981.
3. LAKOTA N.A., RAKHMANOV E.V. and SHVEDOV V.N., Controlling an elastic manipulator on a trajectory. Izv. Akad. Nauk SSSR, Tekhn. kibernetika, No.2, 1980.
4. VERNIGOR V.N., KRAVCHENKO N.F. and POTEEV M.I., On choosing certain constructional parameters of the manipulator arm. Izv. vuzov. Mashinostroenie, No.2, 1982.
5. AKULENKO L.D. and BOLOTNIK N.N., On controlled rotation of an elastic rod. PMM Vol.46, No. 4, 1982.
6. LUR'E A.I., Analytical Mechanics. Moscow, Fizmatgiz, 1961.
7. PANOVKO YA.G. and GUBANOVA I.I., Stability and Oscillations of Elastic Systems. Moscow, NAUKA, 1979.
8. TROITSKII V.A., Optimal Processes of Oscillations of Mechanical Systems. Leningrad, Mashinostroenie, 1976.
9. BUTKOVSKII A.G., Methods of Controlling Systems with Distributed Parameters. Moscow, NAUKA, 1975.
10. FILIPPOV A.P., Oscillations of Deformable Systems. Moscow, Mashinostroenie, 1970.
11. CHERNOUS'KO F.L., AKULENKO L.D. and SOKOLOV B.N., Control of Oscillations, Moscow, NAUKA, 1980.
12. WRIGHT A.D., SMITH C.E., THRESHER R.W. and WANG J.L.C., Vibration modes of centrifugally stiffened beams. Trans. ASME. J. Appl. Mech., Vol.49, No.1, 1982.
13. TRUCKENBRODT A., Bewegungsverhalten und Regulung elastischer Industrieroboter ARME. Z. angew. Math. und Mech., B62, H.4, S. T80-T82, 1982.

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